INTRODUCTION TO
GRAPH DRAWING

Fall 2010

Graph Drawing: Algorithms for the Visualization of Graphs. 1st.
Prentice Hall PTR.
PLANARITY TESTING
**Planarity Testing**

1. Count edges and check Euler's formula
2. Find pieces of G
3. For each piece P that is not a path
test planarity by recursion
4. Compute interlacement graph of the pieces
5. Test if the interlacement graph is bipartite
Planarity Testing

- Graph is planar if and only if all its connected components are planar.
- A connected graph is planar if and only if all its biconnected components are planar.
A graph is *connected* if there is a path between $u$ and $v$ for each pair $(u,v)$ of vertices.

A *cutvertex* in a graph is a vertex whose removal disconnects the graph.

A connected graph with no cutvertices is *biconnected*.

A maximal biconnected subgraph of a graph is biconnected component.
- Decomposing the Graph into connected and biconnected components.
- Our problem will be restricted to testing the planarity of biconnected graphs.
- Use cycle to decompose a biconnected graph into pieces.
PIECE P OF A GRAPH G WITH RESPECT TO PATH C

- The subgraph induced by the edges of path C in a class is called a piece of G with respect to C.

\[ G = (V, E) \]

\[ G' = (V', E') \]
\[ V' \subseteq V \]
\[ E' = E \cap (V' \times V') \]
Let a biconnected graph $G$ contain a cycle $C$

Partition the edges of $G$ not on $C$ into classes:

- Two edges of $G$ are in the same class if there is a path between them that does not contain any vertex of $C$. 
Sub graph induced by edges in a class is called a *piece of G* with respect to C.

- Pieces consisting of a single edge between two vertices of C
- Pieces consisting of a connected graph with at least one vertex not in C
ATTACHMENTS

- Vertices of piece $P$ which are also on cycle $C$ are called attachments of $P$.
- Each piece has at least two attachments. (why?)
Separating cycle

- Cycle $C$ is *separating* if it has at least two pieces.
- And it is *nonseparating* if it has one piece.
LEMMMA 3.4

Let $G$ be a \textit{biconnected} graph and $C$ be a \textit{nonseparating} cycle of $G$ with piece $P$. If $P$ is not a path then $G$ has a separating cycle $C'$ consisting of subpath of $C$ and a path of $P$. 
**Lemma 3.4 Proof**

- Let \( u \) and \( v \) be 2 attachments of \( P \) that are consecutive in the circular ordering.
- Let \( \gamma \) be a subpath of \( C \) between \( u \) and \( v \) without any attachments.
- Since \( P \) is connected there is a path \( \pi \) in \( P \) between \( u \) and \( v \).
- Let \( C' \) be the cycle obtained from \( C \) by replacing \( \gamma \) with \( \pi \).
- Now \( \gamma \) is a piece on \( G \) with respect to \( C' \).
- Let \( e \) be an edge on \( P \) not \( \pi \).
- \( e \) exist Because \( P \) is not a path.
- So there is a piece of \( C' \) other than \( \gamma \) which contains \( e \).
- Thus \( C' \) is a separating cycle in \( G \).
INTERLACEMENT

- Each piece can be drawn either entirely inside or outside of the cycle.
- Interlacing pieces are ones that can’t be drawn on the same side of C without crossing.
INTERLACEMENT GRAPH

- Vertices are pieces on $G$ with respect to cycle $C$
- Edges are pairs of pieces that interlace (can’t reside on the same side of $C$ without crossing).
INTERLACEMENT TO PLANARITY

- If G is planar graph then its interlacement graph must be *bipartite*
**Theorem 3.8**

- A biconnected graph $G$ with cycle $C$ is planar if and only if:
  - For each piece $P$ of $G$ with respect to $C$, the graph obtained by adding $P$ to $C$ is planar.
  - The interlacement graph of the pieces of $G$ with respect to $C$, is bipartite.
Planarity testing

1. Count edges and check Euler's formula
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ALGORITHM PLANARITY TESTING

- Input a biconnected graph $G$ with $n$ vertices and at most $3n-6$ edges, and a separating cycle $C$.

- Output an indication of whether $G$ is planar

1. Compute the pieces of $G$ with respect to $C$. ($O(n)$)
ALGORITHM PLANARITY TESTING

2. For each piece $P$ of $G$ that is not a path:
   - Let $P'$ be the graph obtained by adding $P$ to $C$
   - Let $C'$ be the cycle of $P'$ obtained from $C$ by replacing the portion of $C$ between two consecutive attachments with a path $P$ between them
   - Apply the algorithm recursively to graph $P'$ and cycle $C'$. If $P'$ is nonplanar, return “nonplanar”. ($O(n^2)$)
Algorithm planarity testing

3. Compute the interlacement graph $I$ of the pieces. ($O(n^2)$)
ALGORITHM PLANARITY TESTING

4. Test whether I is bipartite. If I is not bipartite return “nonplanar”. O(n²))

5. Return “planar”.
**OVERALL RUNTIME**

- Each recursive invocation takes $O(n^2)$
- Each recursion is associated with at least one edge of $G$
- There are $O(n)$ edges
- Runtime is $O(n^3)$